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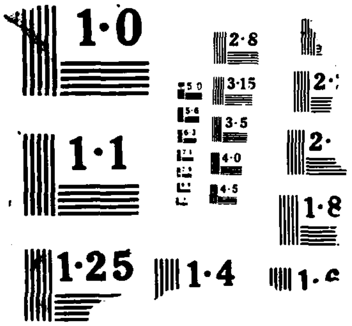
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Technical report 88-4

Embedding Rectangular Grids into Square Grids with Dilation Two*


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and
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*) This research is, in part, supported under ONR Contract N00014-80-C-0455

# EMBEDDING RECTANGULAR GRIDS INTO SQUARE GRIDS WITH DILATION TWO*)

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## ABSTRACT

In this paper, a new technique, the multiple ripple propagation technique, is presented for mapping an $h \times w$ grid into an $w \times h$ grid such that the dilation cost is 2. That is, such that any two neighboring nodes in the first grid are mapped into two nodes in the second grid that are at most distance 2 apart. This technique is then used as a basic tool for mapping any rectangular source grid into a square target grid with the dilation two property preserved. The ratio of the number of nodes in the source grid to the number of nodes in the target grid, called the expansion cost, is shown to be always less than 1.2. This is a noticeable improvement over the previously suggested techniques in which the expansion cost could be bounded by 1.2 only if the dilation cost is allowed to be as high as 18.



*) This work is, in part, supported under ONR contract N00014-80-C-04555.

## 1. Introduction

In this research, we study the problem of squaring up a rectangular grid. That is, embedding an $h \times w$ rectangular grids into an $k \times k$ square grid, where $k \geq \lceil \sqrt{h\ w} \rceil$, and $\lceil\ \rceil$ is the ceiling function. The results of this research may be applied to the VLSI design of highly eccentric circuits that, without squaring up, would have to be laid out in a rectangular area with a height/width ratio very far from unity [5, 8]. They may also be applied to the mapping of rectangular problem domains (as for example finite element grids [3, 6]) into mesh connected architectures [1, 7]. Mapping of rectangular program graphs into hypercube architectures may also benefit from this research. Specifically, it has been shown [4] that this mapping may be accomplished by embedding the graph into a square graph which is, then, mapped easily to the hypercube.

Two measures may be used to estimate the quality of the embedding. The first measure is the expansion cost, $E$, which is the ratio of the number of nodes in the square target grid to the number of nodes in the source rectangular grid. That is $E = k^2/hw$.. The other measure is the dilation cost $D$, which is a measure of the communication penalty that has to be paid due to the squaring up. More specifically, if a link $\lambda$ in the source grid connects two neighboring nodes, say $(i,j)$ and $(i,j+1)$, and these two nodes are mapped to the nodes $(i',j')$ and $(i'+c_i, j'+c_j)$ in the target grid, then the dilation of the edge $\lambda$ after the embedding is defined by $D(\lambda)=|c_i|+|c_j|$. The dilation cost of the embedding is then given by $D = \max_{\lambda} D(\lambda)$.

The best known results for embedding an $h \times w$ grid into the smallest possible $k \times k$ grid are given in [2], where different embedding methods are suggested for different ranges of the eccentricity ratio $\rho = w/h$. Assuming that $h \geq 25$, all the methods suggested in [2] produce embeddings with expansion costs smaller than 1.2, and dilation costs ranging from 2 to 18, depending on the value of $\rho$. Specifically, the dilation cost is less than or equal to 3 if $\rho$ is in one of the ranges (1,2], (10/3,4], (8,9], or $(155,\infty)$. Otherwise, the dilation cost is larger than 5.

In this paper, we first introduce, in Section 2, the multiple ripple propagation technique which may be used to embed an $h \times w$ grid into an $w \times h$ grid with expansion cost 1 and dilation cost 2. This basic technique is then used in Sections 3 and 4 to embed any

rectangular grid with $\rho \leqslant 4$ into a square grid. The idea is to apply the ripple propagation technique to carefully chosen subrectangles of the rectangular grid. For grids with $\rho > 4$, the ripple propagation technique may be combined with the well known technique of folding. This is described and analyzed in Section 5. Finally, in Section 6, we summarize our results and show that, it is always possible to square up any rectangular grid at a dilation cost of 2 and an expansion cost less than 1.2. This is a clear improvement over the results given in [2].

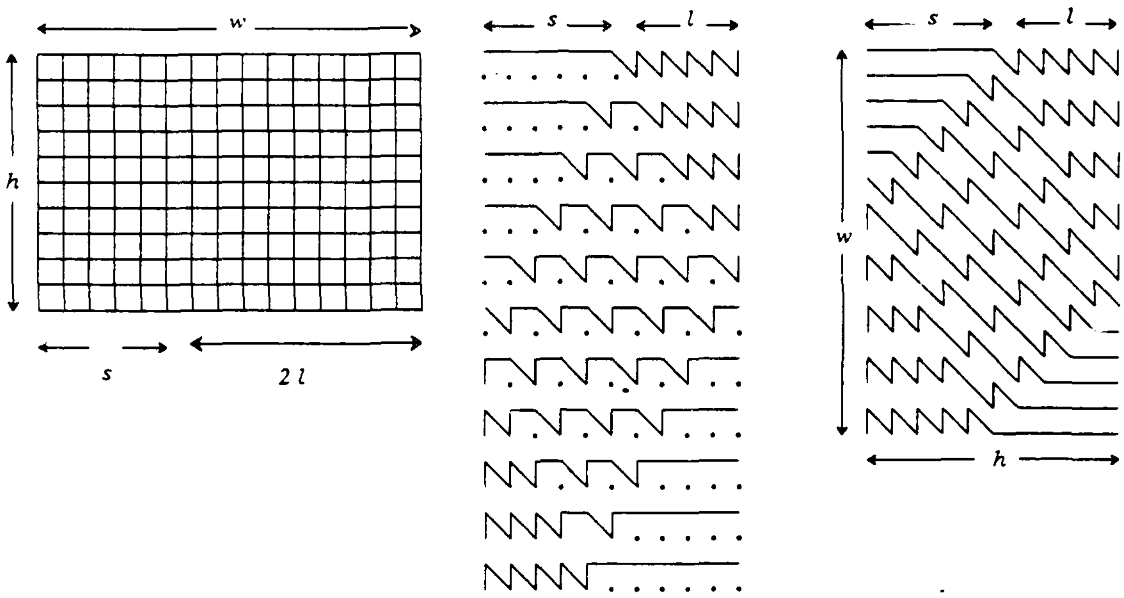
## 2. A multiple ripple propagation technique

The purpose of the technique described in this section is to map an $h \times w$ grid, which satisfy

$$h < w \leqslant 2h \tag{1}$$

into an $w \times h$ grid with unity expansion cost and with dilation cost equal to 2. In order to accomplish that, the $w$ nodes in each row in the original grid should be compressed to occupy only $h$ columns. For this, we let $l = w - h$, and compress $2l$ nodes from each row into $l$ columns by repeated rippling. The remaining $s = w - 2l = 2h - w$ nodes are left uncompressed. In Figure 1(b), we show the grid of Fig 1(a) after compressing each of its rows. As shown in the figure, the positions of the $l$ ripples in each row are chosen as follows: In the first row, the $l$ ripples are grouped to the right, and in the last row, the $l$ ripples are grouped to the left. At each row, one of the ripples that was grouped to the right in the previous row, starts its propagation to the left (moves one column). The propagation of that ripple continues at a rate of one column every row until it can no longer propagate. The propagation of the ripples is very similar to the motion of the legs of a walking worm.

Figure 1(b) is laid out to occupy $w + s$ rows and $h$ columns. However, it may be noticed that $s$ positions in each column is not utilized. This allows for the compression of Fig 1(b) into an $w \times h$ grid which has a dilation cost equal to 2 (see Fig 1(c)). In order to be more formal, we let $F(i,j)=(u(i,j),v(i,j))$ be the function which maps each point $(i,j)$ in the source grid to a corresponding point $(u(i,j),v(i,j))$ in the target grid. For any node $(1,j)$ in the first row of the source grid, the mapping function $F$ is defined as follows:




(a) (b) (c)

Fig 1 - Embedding an 11×16 grid into an 16×11 grid
using multiple ripple propagation

$$u(1,j) = \begin{cases} 1 & \text{for } j=1,\ldots,s \\ 1+rem\left[\dfrac{j-s}{2}\right] & \text{for } j=s+1,\ldots,w \end{cases} \tag{2.a}$$

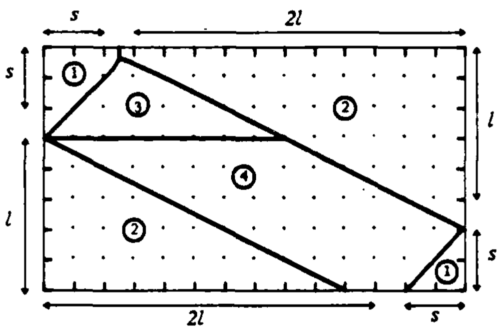
$$v(1,j) = \begin{cases} j & \text{for } j=1,\ldots,s \\ s + \left\lceil \dfrac{j-s}{2} \right\rceil & \text{for } j=s+1,\ldots,w \end{cases} \tag{2.b}$$

Where $rem\,()$ is the remainder of the integer division. The function $F$ may then be defined recursively such that, for any node $(i,j)$ not in the first row, $F(i,j)$ is specified in terms of $F(i-1,j)$. In order to simplify the recursive definition of $F$, we partition the source grid into four regions as shown in Fig 2 and Fig 3, and we use different recursive formulas for different regions. Specifically,

$$u(i,j) = \begin{cases} u(i-1,j)+1 & \text{if } (i,j) \in Region\ 1 \\ u(i-1,j)+2 & \text{if } (i,j) \in Region\ 2 \\ u(i-1,j)+\Delta_{u,3}(i,j) & \text{if } (i,j) \in Region\ 3 \\ u(i-1,j)+\Delta_{u,4}(i,j) & \text{if } (i,j) \in Region\ 4 \end{cases} \tag{3.a}$$

$$v(i,j) = \begin{cases} v(i-1,j) & \text{if } (i,j) \in Region\ 1 \\ v(i-1,j) & \text{if } (i,j) \in Region\ 2 \\ v(i-1,j)-\Delta_{v,3}(i,j) & \text{if } (i,j) \in Region\ 3 \\ v(i-1,j)-\Delta_{v,4}(i,j) & \text{if } (i,j) \in Region\ 4 \end{cases} \tag{3.b}$$

Where, $\Delta_{u,3}$ and $\Delta_{v,3}$ depend on the remainder $r(i,j) = rem\,((j-s+i-1)/3)$. Specifically,




Region 1: $j < s-i+2$ OR $j > w-i+l+1$
Region 2: $j > s+2i-2$ OR $j < 2i-2s-1$
Region 3: $j \geqslant s-i+2$ AND $j \leqslant s+2i-2$ AND $i \leqslant s+1$
Region 4: $j \geqslant 2i-2s-1$ AND $j \leqslant s+2i-2$ AND $i > s+1$ AND $j \leqslant w-i+l+1$.

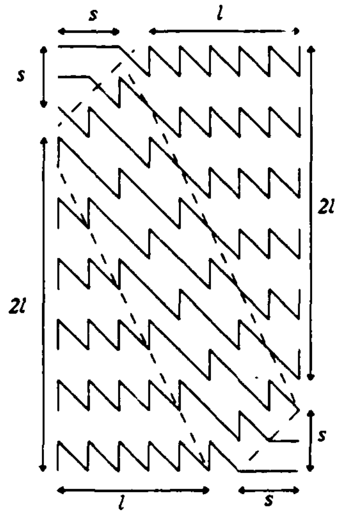
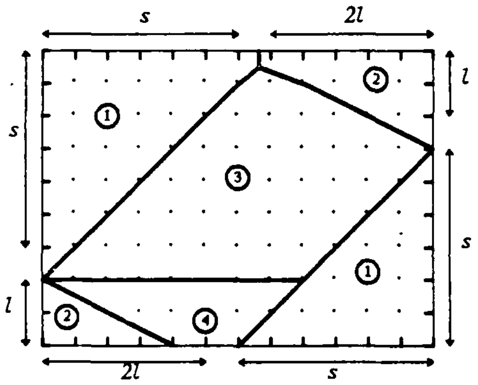



Fig 2 - Partitioning of the source grid for $s \leqslant l$




Region 1: $j < s-i+2$ OR $j > w-i+l+1$
Region 2: $j > s+2i-2$ OR $j < 2i-2s-1$
Region 3: $j \geqslant s-i+2$ AND $j \leqslant s+2i-2$ AND $i \leqslant s+1$ AND $j \leqslant w-i+l+1$.
Region 4: $j \geqslant 2i-2s-1$ AND $j \leqslant w-i+l+1$ AND $i > s+1$.

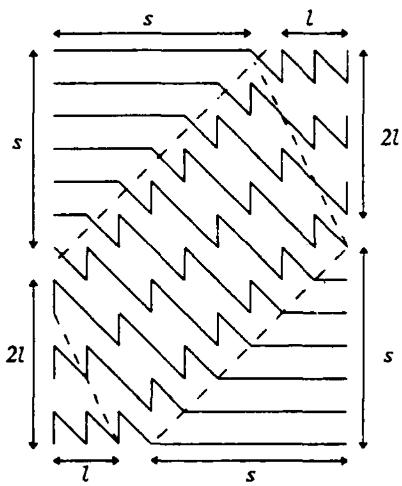



Fig 3 - Partitioning of the source grid for $s > l$

$$\Delta_{u,3} = \begin{cases} 0 & \text{if } r(i,j) = 2 \\ 2 & \textit{otherwise} \end{cases}$$

$$\Delta_{v,3} = \begin{cases} 1 & \text{if } r(i,j) = 2 \\ 0 & \textit{otherwise} \end{cases}$$

Similarly, if $\bar{r}(i,j) = rem((j-2i+2s+2)/3)$, then

$$\Delta_{u,4} = \begin{cases} 0 & \text{if } \bar{r}(i,j) = 2 \\ 2 & \textit{otherwise} \end{cases}$$

$$\Delta_{v,4} = \begin{cases} 1 & \text{if } \bar{r}(i,j) = 2 \\ 0 & \textit{otherwise} \end{cases}$$

Given the above formulas, the following theorem proves that the dilation cost of the mapping $F$ is at most two.

**Theorem:** For any $(i,j)$, where $i,j > 1$, the following is true

$$|u(i,j)-u(i-1,j)| + |v(i,j)-v(i-1,j)| \leqslant 2 \qquad (4.a)$$

and

$$|u(i,j)-u(i,j-1)| + |v(i,j)-v(i,j-1)| \leqslant 2 \qquad (4.b)$$

That is, any two adjacent nodes in the source grid are mapped into two nodes that are not more than a distance two apart in the target grid.

**Proof:** The proof of (4.a) is straight forward if $(i,j)$ is in Region 1 or Region 2. If $(i,j)$ is in region 3, then the left side of (4.a) reduces to $|\Delta_{u,3}(i,j)|+|\Delta_{v,3}(i,j)|$, which is equal to 1 if $r(i,j)$=2, and to 2 otherwise. The case $(i,j) \epsilon$ Region 4 is similar.

To prove (4.b) we use induction on $i$. For $i=1$, the proof is by direct substitution from equations (2). Next, assuming that (4.b) holds for $i-1$, we should show that it also holds for $i$. Again the induction proof is straight forward if $(i,j)$ is in Regions 1 or 2, and is similar if $(i,j)$ is in Regions 3 and 4. For this reason we will consider in the rest of this proof only the case in which $(i,j) \epsilon$ Region 3. For this case, we will prove, by induction, a more restrictive form of (4.b), namely

$$u(i,j)-u(i,j-1) = \begin{cases} -1 & \text{if } r(i,j)=2 \\ 1 & \text{if } r(i,j)=0 \text{ or } 1 \end{cases} \qquad (5.a)$$

$$v(i,j)-v(i,j-1) = \begin{cases} 0 & \text{if } r(i,j)=2 \\ 1 & \text{if } r(i,j)=0 \text{ or } 1 \end{cases} \qquad (5.b)$$

For $i=2$, equation (5) is proved directly from equations (2) and (3). To prove the induction step, we notice that if $(i,j)$ is in Region 3, then $(i,j-1)$ is either in Region 3 or in Region 1. We first assume that $(i,j-1)$ is in Region 3 and use (3) to obtain

$$u(i,j)-u(i,j-1) = u(i-1,j)-u(i-1,j-1) + \Delta_{u,3}(i,j) - \Delta_{u,3}(i,j-1) \qquad (6.a)$$

$$v(i,j)-v(i,j-1) = v(i-1,j)-v(i-1,j-1) - \Delta_{v,3}(i,j) + \Delta_{v,3}(i,j-1) \qquad (6.b)$$

If $r(i,j)$=2, then $\Delta_{u,3}(i,j)$=0 and $\Delta_{u,3}(i,j-1)$=2 because $r(i,j-1)$=1. Also, $r(i-1,j)$=1, which, from the induction hypothesis, gives $u(i-1,j)-u(i-1,j-1)=1$. Therefore, from (6.a) we have $u(i,j)-u(i,j-1)=1+0-2=-1$. Similarly, $\Delta_{v,3}(i,j)=1$, $\Delta_{v,3}(i,j-1)=0$ and $v(i-1,j)-u(i-1,j-1)=1$, from which we obtain

$v(i,j)-v(i,j-1)=1-1+0=0$. The same type of argument applies if $r(i,j)=0$ or 1.

Finally, if $(i,j-1)$ is in Region 1, then $j=s-i+2$ and thus, $r(i,j)=1$. From (3.a), we get $u(i,j)-u(i,j-1) = u(i-1,j)-u(i-1,j-1)+2-1$ and from (3.b) we get $v(i,j)-v(i,j-1) = v(i-1,j)-v(i-1,j-1)$. But both $(i-1,j)$ and $(i-1,j-1)$ are in Region 1, and thus $u(i-1,j)=u(i-1,j-1)$, and $v(i-1,j)=v(i-1,j-1)+1$, which proves (5.a) and (5.b), respectively.[]

The above theorem proves that it is possible to map an $h \times w$ grid, $h < w \leqslant 2h$, exactly into an $w \times h$ grid with dilation cost 2. It is also possible to concatenate the $w \times h$ target grid with its symmetric image (reflected across the line $v=h$) to obtain an exact embedding of an $h \times 2w$ grid into an $w \times 2h$ grid with dilation cost 2. Along the same line of thinking, an $h \times 2w+1$ source grid may be divided into an $h \times w+1$ and an $h \times w$ subgrids. These two subgrids may then be embedded into an $w+1 \times h$ and an $w \times h$ grids, respectively, and by concatenating the former with the symmetric image of the latter, we may obtain an $w+1 \times 2h$ target grid. The dilation cost at the line of concatenation may be shown to be at most two. Grid concatenations of the type described here will be used repeatedly and tacitly in the rest of this paper.

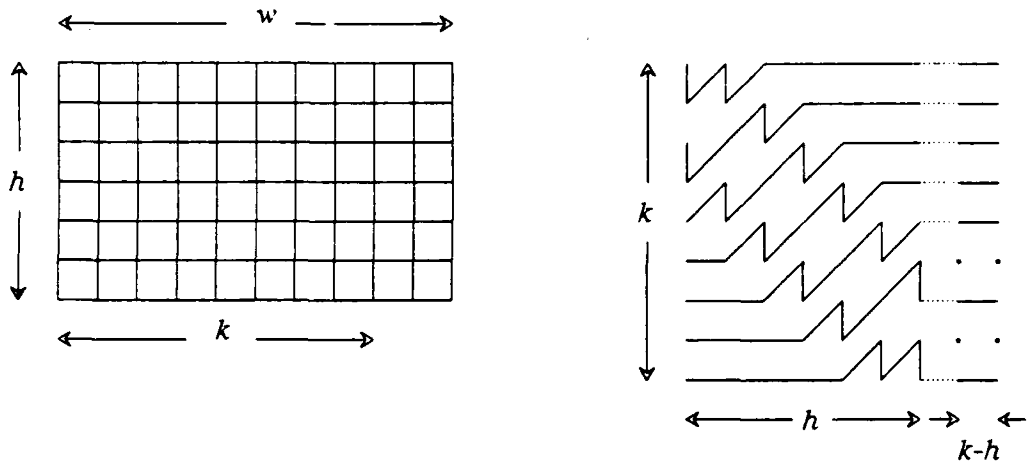
In the following sections, we apply the above technique to our original problem of mapping an $h \times \rho h$ rectangular grid ($\rho$ is assumed to be greater than unity), into a square grid. First, two basic methods are introduced for grids with $\rho \leqslant 4$. These methods are then combined with folding and applied effectively to the embedding of any grid with $\rho > 4$.

## 3. The method of exact row fitting.

Let $k$, $k > h$ be the dimension of the square grid (called the target grid) onto which the given $h \times \rho h$ grid (called the source grid) is to be mapped. Of course, it is desirable to chose the smallest possible $k$ in order to minimize the expansion cost $E = k^2/\rho h^2$. Given such a $k$, the method of exact row fitting assumes that the right most $h \times k$ subgrid of the source grid may be mapped exactly into the $k \times h$ right most subgrid of the target grid (see Fig 4). This is possible if and only if condition (1) is satisfied. That is

$$h \leqslant k \leqslant 2h \tag{7.a}$$

Moreover, if




(a) (b)

Fig 4 - Embedding an 7×11 grid into an 9×9 grid
using the method of exact row fitting

$$k - h \geqslant \rho h - k \tag{7.b}$$

that is, the number of columns, $k-h$, remaining in the target grid is at least equal to the number of columns, $\rho h - k$, remaining in the source grid, then these columns may be mapped in a trivial way with dilation cost 2. In other words, the mapping may be completed with dilation cost 2 provided that *the size of the target grid, $k$, satisfies the conditions* (7.a/b).

The solution of the inequalities (7.a/b) may be found by, first, computing the minimum $k$ that satisfies (7.b), and then checking that this value is consistent with (7.a). Specifically, (7.b) is satisfied if

$$k = K_r = \left\lceil \frac{\rho+1}{2} h \right\rceil \tag{8}$$

It is straight forward to check that the value of $k$ given by (8) satisfies (7.a) if $\rho \leqslant 3 - 1/h$. Hence, the method of exact row fitting may be applied only if $\rho \leqslant 3 - 1/h$. Noting that $K_r \leqslant ((\rho+1)h+1)/2$, we may obtain an upper bound on the expansion cost of the resulting embedding. Namely,

$$E_r = \frac{K_r^2}{\rho} \leqslant E_{r,max} = \frac{(\rho+1+\frac{1}{h})^2}{4\rho} \tag{9}$$

The value of $E_r$ increases monotonically with $\rho$ for $\rho > 1+1/h$, and hence may exceed 1.2 for large values of $\rho$. For example, assuming $h=12$, then $E_r > 1.2$ if $\rho \geqslant 2.06$. Moreover,

if $\rho > 3 - 1/h$, the method may not be applied. In these cases, the method of exact column fitting, described in the following section, can be used.

## 4. The method of exact column fitting

The embedding technique used in this section is based on the vertical disection of both the source and the target grids, each into two subgrids which are as equal as possible. Each of the source subgrids is then embedded into the corresponding target subgrid in a way that ensures that all the columns of the target grid are efficiently used. In order to deal with the case of $\rho h$ being an odd integer, the number of columns in the two source subgrids is taken to be $\lceil \rho h/2 \rceil$ and $\lfloor \rho h/2 \rfloor$, respectively, where $\lfloor\, \rfloor$ is the floor function. For the same reason, the number of columns in the target subgrid is divided into $\lceil k/2 \rceil$ and $\lfloor k/2 \rfloor$ columns, respectively (see Fig 5).

The optimal size, $k = K_c$, of the target grid should be determined by the embedding of $h \times \lceil \rho h/2 \rceil \rightarrow k \times \lceil k/2 \rceil$ or the embedding $h \times \lfloor \rho h/2 \rfloor \rightarrow k \times \lfloor k/2 \rfloor$, whichever gives a more strict condition on $k$. It turns out that the latter embedding is more restrictive than the former, and hence, should be used to derive $k$. In the remaining of this section, we will denote the $h \times \lfloor \rho h/2 \rfloor$ grid by $G_s$, and the $k \times \lfloor k/2 \rfloor$ grid by $G_t$, and we will describe an embedding of $G_s$ into $G_t$. The embedding of the other half of the source grid (the $h \times \lceil \rho h/2 \rceil$ subgrid) into the other half of the target grid (the $k \times \lceil k/2 \rceil$ subgrid) may be accomplished in a similar fashion.

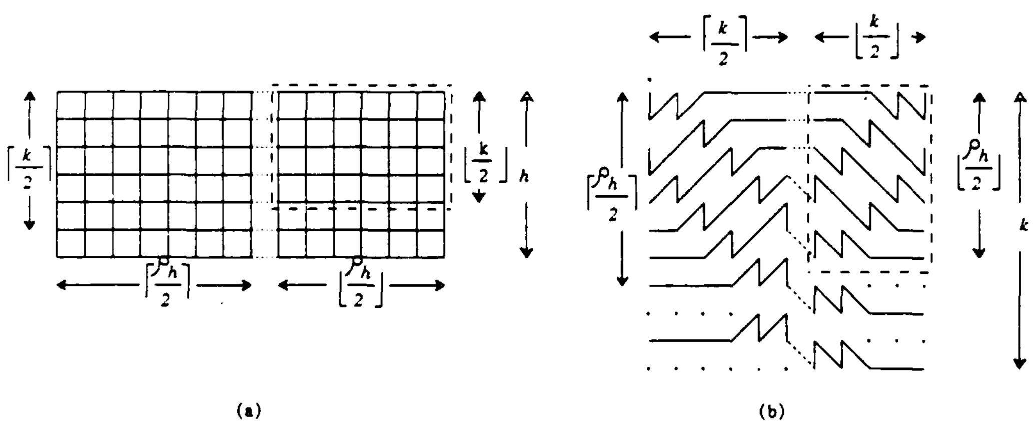



Fig 5 - Embedding an 7×15 grid into an 11×11 grid using exact column fitting

Consider the upper $\lfloor k/2 \rfloor \times \lfloor \rho h/2 \rfloor$ subgrid of $G_s$, and embed it into the upper $\lfloor \rho h/2 \rfloor \times \lfloor k/2 \rfloor$ subgrid of $G_t$ using the rippling propagation technique of Section 2. In order to accomplish this embedding we should have

$$\lfloor k/2 \rfloor \leqslant h \tag{10.a}$$

and condition (1) should be satisfied, namely

$$\lfloor k/2 \rfloor \leqslant \lfloor \rho h/2 \rfloor \leqslant 2 \lfloor k/2 \rfloor \tag{10.b}$$

With this, each of the remaining $h - \lfloor k/2 \rfloor$ rows in $G_s$ may then be compressed to have the same pattern as the last row of the $\lfloor \rho h/2 \rfloor \times \lfloor k/2 \rfloor \rightarrow \lfloor k/2 \rfloor \times \lfloor \rho h/2 \rfloor$ embedding. This results in a dilation cost equal to 2 and requires $2(h - \lfloor k/2 \rfloor)$ additional rows in $G_t$. Thus, the following should be satisfied

$$k - \lfloor \rho h/2 \rfloor \leqslant 2\,(h - \lfloor k/2 \rfloor) \tag{10.c}$$

Noting that $(x+1)/2 \geqslant \lfloor x/2 \rfloor \geqslant (x-1)/2$, we may calculate the minimum value of $k$ which always satisfies (10.c). Namely,

$$k = K_c = \left\lceil \frac{\rho h/2 + 2h + 1}{2} \right\rceil \tag{11}$$

This value of $K_c$ satisfies the conditions (10.a) and (10.b) as long as $\rho \leqslant 4$.

With the value of $k$ given by (11), the two halves of the source grid may be successfully embedded into the two halves of the target grid with dilation cost 2. Noting that $\lfloor x/4 \rfloor \leqslant (x+3)/4$, it is possible to bound the expansion cost of the embedding as follows:

$$E_c \leqslant E_{c,max} = \frac{(\rho + 4 + \frac{5}{h})^2}{16\rho} \tag{12}$$

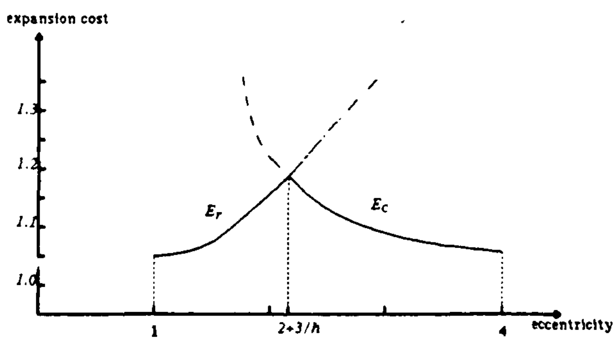



Fig 6 - Expansion cost Vs eccentricity for $1 < \rho \leqslant 4$ ($h = 20$)

For $1 < \rho \leqslant 4$, the value of $E_c$ is monotonically decreasing with $\rho$, which suggests the use of the method of fitting columns whenever the method of fitting rows fails to satisfy

$E_r \leqslant 1.2$ (see Fig 6). The critical value of $\rho$ that determines which of the two methods has a smaller expansion cost may be found by solving $E_{r,max} = E_{c,max}$. From (9) and (12), this gives $\rho = 2+3/h$. The expansion cost at this value of $\rho$ is $(3h+4)^2/(4h(2h+3))$, which is always smaller than 1.2 if $h > 17$.

Hence, for $1 < \rho \leqslant 4$, the most efficient embedding method depends on the value of $\rho$. Specifically, if $\rho \leqslant 2+3/h$, then the methods of exact row fitting should be used, otherwise, the method of exact column fitting should be used. For values of $\rho$ larger than four, the above methods can be combined with the known method of folding [2] as described in the next section.

## 5. Combining ripple propagation with folding.

If $\rho = (q+1)^2$ for some integer $q \geqslant 1$, then the source grid may be folded $q+1$ times to fit exactly an $(q+1)h \times (q+1)h$ target grid. In fact, it is easy to show that if

$$\frac{(q+1)^2}{1.2} < \rho \leqslant (q+1)^2 \tag{13.a}$$

for some integer $q \geqslant 1$, then folding the source grid into an $(q+1)h \times (q+1)h$ target grid will result in an expansion cost less than 1.2. In Fig 7, we illustrate the technique of folding by an example. As clear from this figure, successive tracks (a track consists of $h$ consecutive rows of the target grid) are joined by two $h \times h$ corner tiles that guarantee a dilation cost equal to two.

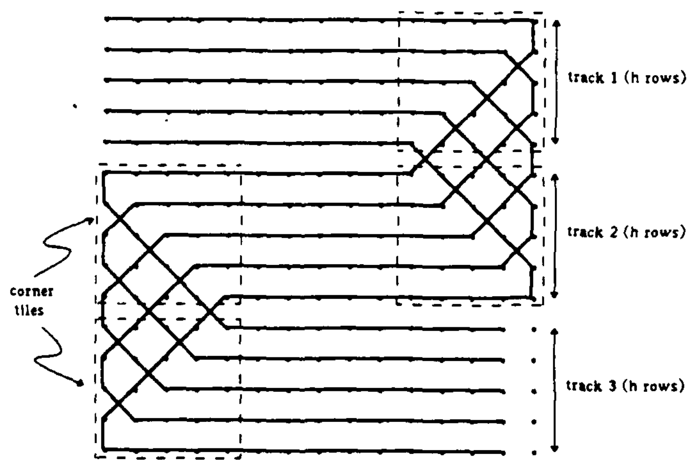



Fig 7 - Folding an 5×40 grid into an 15×15 grid

As described above, folding may result in few unused columns in the last track of the

target grid, and condition (13.a) limits the number of these unused columns. It is also possible to apply folding and leave some rows of the target grid unused. More precisely, if the eccentricity of the source grid satisfies

$$q^2 \leqslant \rho \leqslant 1.2q^2 \tag{13.b}$$

then, it is possible to fold this grid into an $\rho h/q \times \rho h/q$ target grid. This will leave $(rho - q^2)h/q$ unused rows in the target grid, and condition (13.b) will guarantee that the number of unused rows does not exceed $0.2qh$. Thus, the expansion cost will be less than 1.2.

### 5.1. Combining folding with exact row fitting

Consider an $h \times \rho h$ source grid which satisfies $1.2q^2 < \rho < (q+1)^2/1.2$. Clearly, folding this grid into a square grid is too expensive (expansion cost larger than 1.2) because neither (13.a) nor (13.b) is satisfied. In this section, we introduce a method which combines folding and exact row fitting. This method will be denoted by FR. In order to describe the FR method we assume that the source grid is to be embedded into a target grid of size $k$, where $k$ satisfies $qh \leqslant k \leqslant (q+1)h$. The embedding starts by folding the source grid into the target grid $q$ times as shown in Fig 8(a). Clearly, the right most $h \times (\rho h - qk)$ subgrid of the source grid will not fit into the target grid, and the last $k - qh$ rows of the target grid will be unused. The idea is to consider the last track resulting from the folding (an $h \times \rho h - (q-1)k$ grid denoted by $G_s$), and to squeeze it into an $k - (q-1)h \times k$ grid (denoted by $G_t$) that fits the target grid.

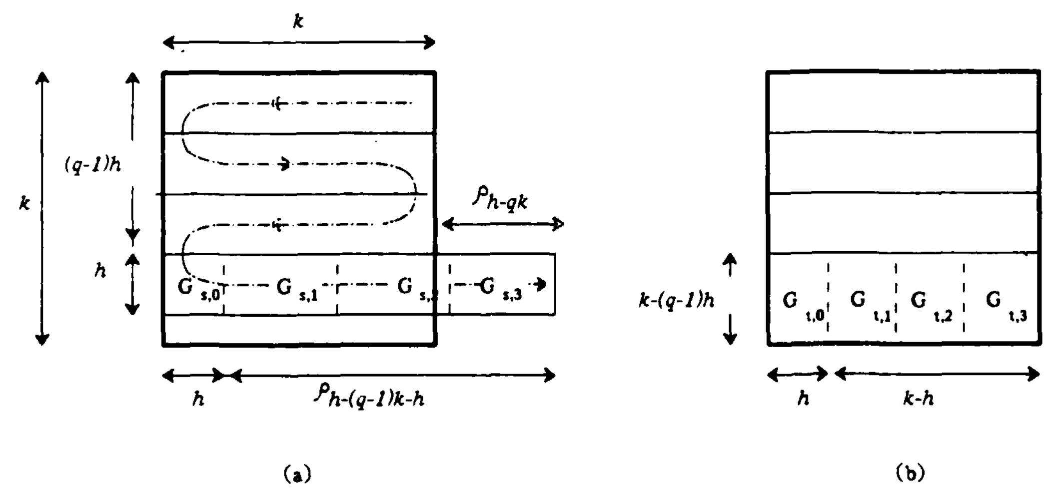



Fig 8 - Combining folding with exact row fitting ($q$ =4, and $P$ =4)

The squeeze is performed by partitioning $G_s$ vertically into $P$ subgrids $G_{s,0}, \cdots, G_{s,P-1}$, and partitioning $G_t$ vertically into $P$ subgrids, $G_{t,0}, \cdots, G_{t,P-1}$, and then mapping each $G_{s,i}$ into the corresponding $G_{t,i}$. The partitioning of $G_s$ is such that $G_{s,0}$, is an $h \times h$ grid and each remaining $G_{s,i}$, $i=1,\ldots,P-1$ is an $h \times \lfloor(\rho h-(q-1)k-h)/(P-1)\rfloor$ grid. Note that if $\rho h-(q-1)k-h$ does not divide $P-1$ then $G_{s,P-1}$ will have few empty columns. Similarly, the partition of $G_t$ is such that $G_{t,0}$ is an $k-(q-1)h \times h$ grid and each of the remaining $G_{t,i}$, $i=1,\ldots,P-1$ is an $k-(q-1)h \times \lfloor(k-h)/(P-1)\rfloor$ grid.

The method of exact row fitting introduced in Section 3 is used to map each $G_{s,i}$, $i=1,\ldots,q-1$, into the corresponding $G_{t,i}$. As for the mapping $G_{s,0} \rightarrow G_{t,0}$, it should ensure that the transition from track $q-1$ to track $q$ does not increase the dilation cost beyond two. This may be accomplished by expanding the $h \times h$ corner tiles (see Fig 7) into an $k-(q-1)h \times h$ pattern that fits $G_{t,0}$ such that the distribution of the $h$ nodes in the last column of $G_{t,0}$ is similar to the distribution of the $h$ nodes in the first column of $G_{t,1}$. It may be shown that such an expansion is always possible with dilation cost not exceeding two.

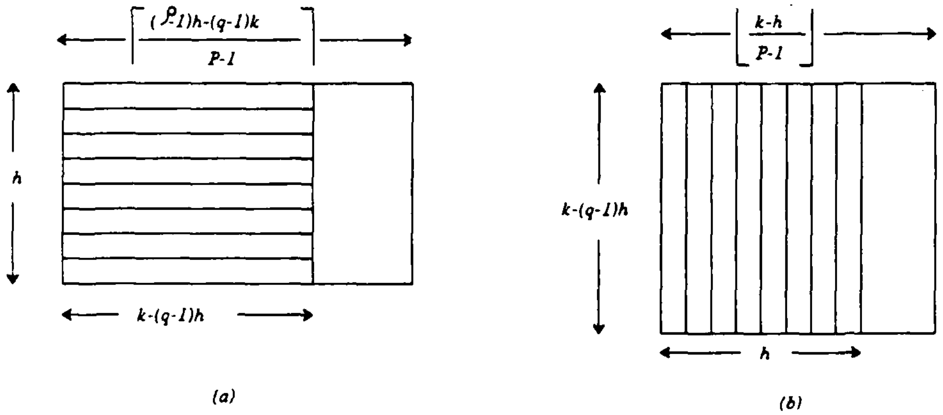



Fig 9 - Mapping $G_{s,i}$ into $G_{t,i}$ in the FR method

In order to compute the optimum size $k$ of the target grid, we follow the same reasoning as in section 3. Specifically, the method of exact row fitting may be used for mapping any $G_{s,i}$ into the corresponding $G_{t,i}$. For this, the following conditions should be satisfied (refer to Fig 9):

$$h \leqslant \lfloor \frac{k-h}{P-1} \rfloor \qquad (14.a)$$

$$h \leqslant k - (q-1)h \leqslant 2h \tag{14.b}$$

$$\lfloor \frac{k-h}{P-1} \rfloor - h \geqslant \lfloor \frac{(\rho-1)h - (q-1)k}{P-1} \rfloor - (k-(q-1)h) \tag{14.c}$$

In order to solve the above system of inequalities, we first find the minimum value of $k$ which always satisfies (14.c). This value is:

$$k = K_{fr} = \lceil \frac{(\rho + q(P-1))h + 2(P-2)}{q+P-1} \rceil \tag{15}$$

By substituting (15) in (14.a) and (14.b), we conclude that these two conditions are satisfied, respectively if

$$\rho \geqslant P^2 + q - P + \frac{(P+q-3)(P-2)}{h} \tag{16.a}$$

and

$$\rho \leqslant q(q+1) + P - 1 - \frac{3P+q-6}{h} \tag{16.b}$$

Hence, given any rectangular grid, the method may be used if there exists a $P$ that satisfies (16). The number of partitions $P$ also affects the expansion cost. More precisely, from (15), we find that $K_{fr} \leqslant ((\rho h + q(P-1)h + 3P + q - 6)/(q+P-1)$, which may be used to bound the expansion cost by

$$E_{fr} \leqslant E_{fr,max} = \frac{(\rho + q(P-1) + (3P+q-6)/h)^2}{(P+q-1)^2 \rho} \tag{17}$$

The derivative $\partial E_{fr,max}/\partial P$ is negative for $\rho \geqslant q^2$ which means that, from the point of view of minimizing $E_{fr}$, it is advantageous to find the maximum $P$ which satisfies (16). For $P > q$, the two conditions (16.a) and (16.b) may not be satisfied simultaneously. If, however, $\rho$ is in the range

$$q^2 + \frac{(q-2)(2q-3)}{h} \leqslant \rho \leqslant (q+1)^2 - 2 - \frac{4q-6}{h}, \tag{18}$$

then, (16.a/b) are satisfied for $P = q$, and hence the embedding may be completed with $q$ partitions in a target grid whose size is given by (15). The maximum expansion cost may then be found by substituting $P = q$ in (17) to obtain

$$E_{fr} \leqslant E_{fr,max} = \frac{(\rho + q(q-1) + (4q-6)/h)^2}{(2q-1)^2 \rho} \tag{19}$$

In Section 6, it will be shown that, for $q \geqslant 3$, $E_{fr}$ is smaller than 1.2 for any $\rho$ in the range specified by (18), and that, outside that range, $\rho$ satisfies (13.a) or (13.b), which means that folding may be used with expansion cost less than 1.2. The case $q = 2$, however,

is slightly more complicated. For instance, if $h \geqslant 20$, then, folding is too expensive in the range $5.85 \leqslant \rho \leqslant 7.5$ (expansion cost is larger than 1.2). Also in that range, the FR method either does not apply (if $\rho > 6.9$) or gives $E_{fr} > 1.2$ (if $5.85 < \rho < 6.9$). In this case, combining folding with the method of exact column fitting (the FC method) turns out to be useful. Although we only need this combination for $q = 2$, the FC method will be described in the next section for general $q$. The reason for doing so is that for $q \geqslant 3$, although both the FC and the FR methods realize an expansion cost less than 1.2, it will be shown that the FC method gives better results than the FR method for some subranges of $\rho$.

### 5.2. Combining folding with exact column fitting

In this method, denoted from now on by FC, the source grid is folded into the target grid as described in the previous section, and also each of $G_s$ and $G_t$ is partitioned into $P$ subgrid. The FC method is different from the FR method in that each subgrid $G_{s,i}$, $i = 1, \ldots, P-1$, is mapped into the corresponding $G_{t,i}$ using the method of exact column fitting rather than exact row fitting.

The conditions that have to be satisfied in order to map $G_{s,i}$ into $G_{t,i}$ using exact column fitting are analogous to the conditions (10.a/b/c) of Section 4. Specifically, these conditions are (refer to Fig 10):

$$\left\lfloor \frac{(\rho-1)h-(q-1)k}{P-1} \right\rfloor \leqslant k-(q-1)h \tag{20.a}$$

$$\left\lfloor \frac{k-h}{P-1} \right\rfloor \leqslant \left\lfloor \frac{(\rho-1)h-(q-1)k}{P-1} \right\rfloor \leqslant 2\left\lfloor \frac{k-h}{P-1} \right\rfloor \tag{20.b}$$

$$k-(q-1)h - \left\lfloor \frac{(\rho-1)h-(q-1)k}{P-1} \right\rfloor \geqslant 2\left(h - \left\lfloor \frac{k-h}{P-1} \right\rfloor\right. \tag{20.c}$$

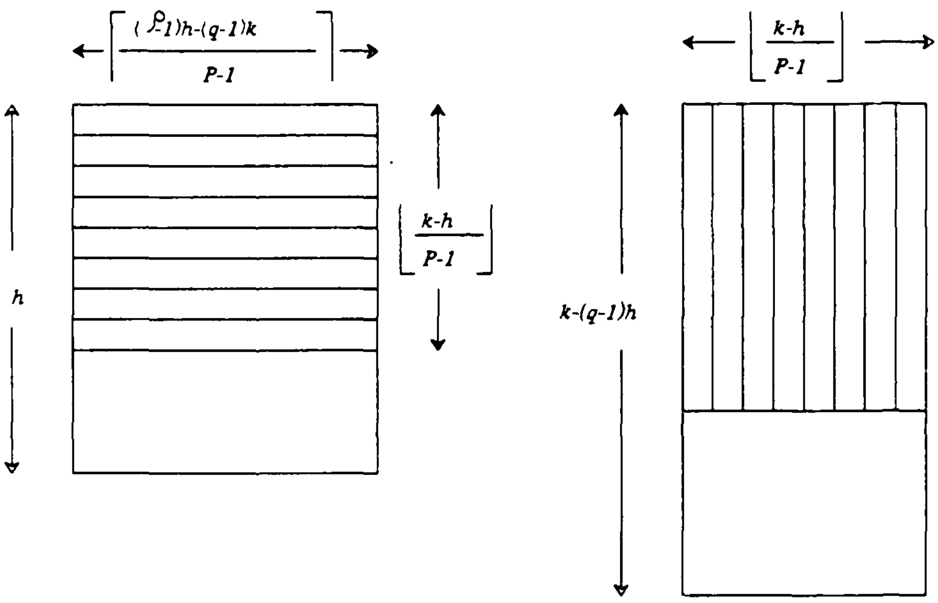
The same technique that was used in the last sections is applied to the solution of the above inequalities. From (20.c), the minimum size of the target grid is found to be

$$k = K_{fc} = \left\lfloor \frac{(\rho+qP+P-q)h + 3(P-2)}{P+q} \right\rfloor \tag{21}$$

By substituting (21) in (20.a) we obtain the condition

$$\rho \leqslant P^2-P+q + (P+q-3)(P-2)/h$$

which may be satisfied only if $P > q$. Also, by using (21) to compute the expansion cost $E_{fc}$, and then differentiating the resulting formula, we find that $\partial E_{fc} / \partial P$ is positive for $\rho \leqslant P^2-P+q$. This means that using $P = q+1$ partitions will give the best expansion cost.




(a) (b)

Fig 10 - Mapping $G_{s,i}$ into $G_{t,i}$ in the FC method

Now using $P=q+1$ in (21), and substituting the result in (20.a) and (20.b), we find that these conditions are satisfied if $\rho$ lies in the following range

$$q^2+1+\frac{3(q-1)}{h} \leqslant \rho \leqslant (q+1)^2-1-\frac{(q-1)^2}{qh} \tag{22}$$

That is, the FC method may be applied if $\rho$ satisfies (22). The expansion cost may then be computed from (21) with $P=q+1$. The upper bound on this cost is given by

$$E_{fc} \leqslant E_{fc,max} = \frac{(\rho+q(q+1)+1+(5q-3)/h)^2}{(2q+1)^2\rho} \tag{23}$$

**6. Discussion and conclusion**

Given an $h \times \rho h$ source grid, let $q$ be the integer that satisfies $q^2 \leqslant \rho < (q+1)^2$. For $q=1$, it has been shown in Sections 3 and 4 that the mapping of the source grid into a square rectangular grid may be accomplished by using the method of exact row fitting if $\rho \leqslant 2+3/h$, or the method of exact column fitting if $\rho > 2+3/h$. In both cases, the expansion cost is proven to be less than 1.2.

For, $q \geqslant 2$, the FR or the FC methods described in Section 5 may be applied provided that $\rho_1 < \rho < \rho_2$, where the critical values $\rho_1$ and $\rho_2$ are specified from (18) and (22).

Namely,

$$\rho_1 = q^2 + \frac{(q-2)(2q-3)}{h} \tag{24.a}$$

$$\rho_2 = (q+1)^2 - 1 - \frac{(q-1)^2}{qh} \tag{24.b}$$

In order to determine which of the two methods gives a smaller expansion cost, we notice from (19) and (23) that, for $\rho > 0$, $E_{fr,max}$ and $E_{fc,max}$ intersect at only one point, namely

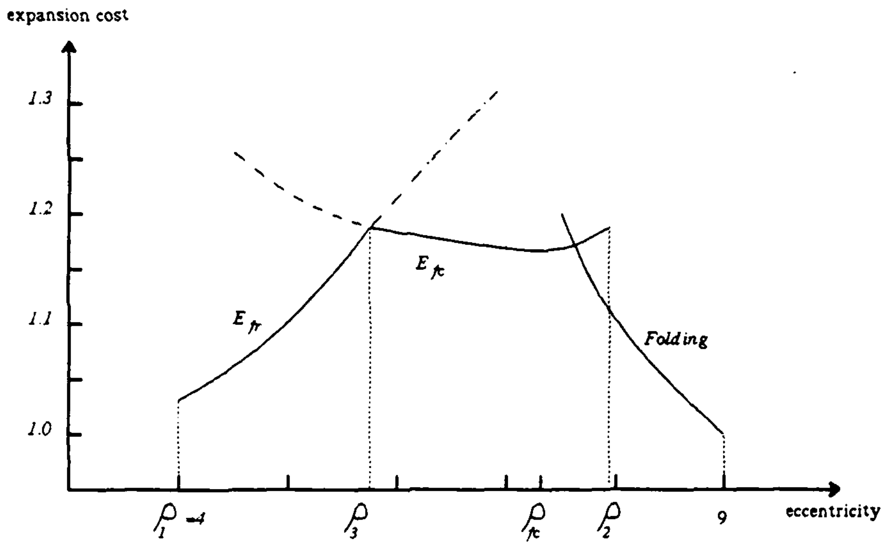
$$\rho_3 = q(q+1) - \frac{1}{2} + \frac{2q^2-3q+9}{2h} \tag{24.c}$$

We also observe that both $E_{fr,max}$ and $E_{fc,max}$ are of the form $f(\rho)=(\rho+a)^2/(b\rho)$, for some constants $a$ and $b$. Given that, for $\rho>0$, the function $f$ has only one local minima at $\rho=a$, we may determine that $E_{fr,max}$ has its local minima at $\rho_{fr}=q(q-1)+(4q-6)/h$, and $E_{fc,max}$ has its minima at $\rho_{fc}=q(q+1)+1+(5q-3)/h$. Clearly, $\rho_{fr}$ is smaller than $\rho_1$ for $h>4$, and $\rho_{fc}$ lies between $\rho_1$ and $\rho_2$. This leads to the conclusion that $E_{fc,max}<E_{fr,max}$ if $\rho>\rho_3$ and $E_{fc,max}>E_{fr,max}$ if $\rho<\rho_3$. In Figure 11, both $E_{fc,max}$ and $E_{fr,max}$ are plotted for $q=2$, $h=20$, and for $q=4$, $h=10$.

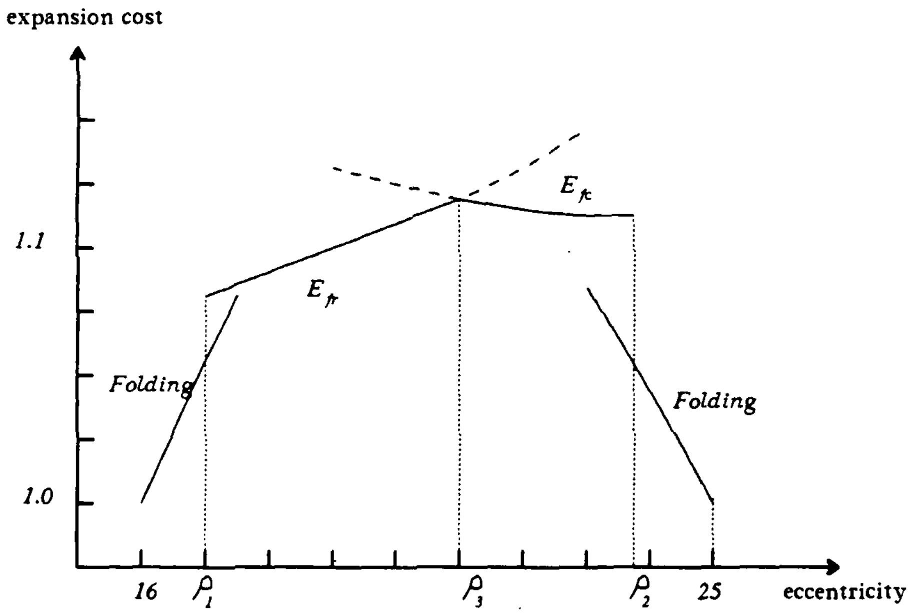
Hence, if $\rho$ lies between $\rho_1$ and $\rho_3$, the FR method is recommended, and if $\rho$ lies between $\rho_3$ and $\rho_2$, then the FC method is recommended. If this strategy is applied, then the largest expansion cost occurs at either $\rho=\rho_2$ or $\rho=\rho_3$. By direct substitution of (24.b) and (24.c) into (23), and after simple algebraic manipulation, it may be shown that, for $h \geqslant 18$, the value of $E_{fc,max}$ is less than 1.2 at $\rho_2$ and $\rho_3$.

Neither the FR method nor the FC method may be applied if $\rho$ is less than $\rho_1$ or larger than $\rho_2$. However, in these two cases, the expansion cost resulting from simple folding is low because $\rho$ is close enough to $q^2$ and $(q+1)^2$, respectively. In fact, if $\rho \leqslant \rho_1$, then $\rho<q^2(1+2/h)$, which satisfies (13.b) if $h>10$. Also, if $\rho \geqslant \rho_2$, then $\rho \geqslant (q+1)^2(1-1/qh)-1$, which satisfies (13.a) if $h>10$. In other words, the application of simple folding in these two regions will result in an expansion cost less than 1.2.

In brief, new techniques have been presented and analyzed in this paper, for embedding an $h \times \rho h$ rectangular grid into a square grid with dilation cost equal to two. The most appropriate technique for a given grid have been shown to depend on the size of that grid, that is on $h$ and $\rho$. By adhering to the selection strategy suggested in the paper, the expan-




(a) $q=2,\ h=20$




(b) $q=4,\ h=10$

Fig 11 - Expansion cost for $q^2 \leqslant \rho < (q+1)^2$

sion cost is guaranteed to be smaller than 1.2 if $h$ is larger than or equal to 18.

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